Lecture Notes, January 9, 2012

Why do we study mathematical general equilibrium theory?

To give economics a logical foundation as sound as algebra or Euclidean geometry.

Prof. Hugo Sonnenschein (2005):

In 1954, referring to the first and second theorems of classical welfare economics, Gerard [Debreu] wrote 'The contents of both Theorems ... are old beliefs in economics. Arrow and Debreu have recently treated these questions with techniques permitting proofs.' This statement is precisely correct; once there were beliefs, now there was knowledge.

But more was at stake. Great scholars change the way that we think about the world, and about what and who we are. The Arrow-Debreu model, as communicated in [Debreu's] *Theory of Value* changed basic thinking, and it quickly became the standard model of price theory. It is the 'benchmark' model in Finance, International Trade, Public Finance, Transportation, and even macroeconomics. ... In rather short order it was no longer 'as it is' in Marshall, Hicks, and Samuelson; rather it became 'as it is' in *Theory of Value*.

General Equilibrium Theory

Market-clearing prices and allocation of several goods, taking into account cross-market interactions.

Existence of Equilibrium
(Pareto) Efficiency of Allocation
Decentralization of decision-making

The Edgeworth Box

2 person, 2 good, pure exchange economy

Fixed positive quantities of X and Y, and two households, 1 and 2.

Household 1 is endowed with $\bar{X}^1$ of good X and $\bar{Y}^1$ of good Y, utility function $U^1(X^1, Y^1)$. Household 2 is endowed with $\bar{X}^2$ of good X and $\bar{Y}^2$ of good Y, utility function $U^2(X^2, Y^2)$

$$X^1 + X^2 = \bar{X}^1 + \bar{X}^2 \equiv \bar{X},$$
$$Y^1 + Y^2 = \bar{Y}^1 + \bar{Y}^2 \equiv \bar{Y} \quad .$$

Each point in the Edgeworth box represents an attainable choice of X^1 and X^2 , Y^1 and Y^2 .

1's origin is at the southwest corner; 1's consumption increases as the allocation point moves in a northeast direction; 2's increases as the allocation point moves in a southwest direction. Superimpose indifference curves on the Edgeworth Box.

Competitive Equilibrium

(p^o_x, p^o_y) so that (X^{o1}, Y^{o1}) maximizes $U^1(X^1, Y^1)$ subject to
$(p^o_x, p^o_y)\cdot(X^1, Y^1) \leq (p^o_x, p^o_y)\cdot(\bar{X}^1, \bar{Y}^1)$ and
(X^{o2}, Y^{o2}) maximizes $U^2(X^2, Y^2)$ subject to
$(p^o_x, p^o_y)\cdot(X^1, Y^1) \leq (p^o_x, p^o_y)\cdot(\bar{X}^2, \bar{Y}^2)$, and
$(X^{o1}, Y^{o1}) + (X^{o2}, Y^{o2}) = (\bar{X}^1, \bar{Y}^1) + (\bar{X}^2, \bar{Y}^2)$
or $(X^{o1}, Y^{o1}) + (X^{o2}, Y^{o2}) \leq (\bar{X}^1, \bar{Y}^1) + (\bar{X}^2, \bar{Y}^2)$, where the inequality holds co-ordinatewise and any good for which there is a strict inequality has a price of 0.

Pareto efficiency:

An allocation is Pareto efficient if all of the opportunities for mutually desirable reallocation have been fully used. The allocation is Pareto efficient if there is no available reallocation that can improve the utility level of one household while not reducing the utility of any household.

Tangency of 1 and 2's indifference curves : Pareto efficient allocations.

Pareto efficient allocation:
$(X^{o1}, Y^{o1}), (X^{o2}, Y^{o2})$ maximizes

$U^1(X^1, Y^1)$ subject to

$U^2(X^2, Y^2) \geq U^{o2}$ (typically equality will hold and $U^{o2}=U^2(X^{o2}, Y^{o2})$) and subject to the resource constraints

$$X^1 + X^2 = \bar{X}^1 + \bar{X}^2 \equiv \bar{X}$$

$$Y^1 + Y^2 = \bar{Y}^1 + \bar{Y}^2 \equiv \bar{Y} \quad .$$

Equivalently, $X^2 = \bar{X} - X^1$, $Y^2 = \bar{Y} - Y^1$

Solving for Pareto efficiency (Assuming differentiability and an interior solution):

Lagrangian

$$L \equiv U^1(X^1, Y^1) + \lambda[U^2(\bar{X} - X^1, \bar{Y} - Y^1) - U^{o2}]$$

$$\frac{\partial L}{\partial X^1} = \frac{\partial U^1}{\partial X^1} - \lambda\frac{\partial U^2}{\partial X^2} = 0$$

$$\frac{\partial L}{\partial Y^1} = \frac{\partial U^1}{\partial Y^1} - \lambda\frac{\partial U^2}{\partial Y^2} = 0$$

$$\frac{\partial L}{\partial \lambda} = U^2(X^2, Y^2) - U^{o2} = 0$$

This gives us then the condition

$$MRS^1_{xy} = \frac{\frac{\partial U^1}{\partial X^1}}{\frac{\partial U^1}{\partial Y^1}} = \frac{\frac{\partial U^2}{\partial X^2}}{\frac{\partial U^2}{\partial Y^2}} = MRS^2_{xy} \text{ or equivalently}$$

$$MRS^1_{xy} = \frac{\partial Y^1}{\partial X^1}\Big|_{U^1=\text{constant}} = \frac{\partial Y^2}{\partial X^2}\Big|_{U^2=\text{constant}} = MRS^2_{xy}$$

Pareto efficient allocation in the Edgeworth box: the slope of 2's indifference curve at an efficient allocation will equal the slope of 1's indifference curve; the points of tangency of the two curves.

contract curve = individually rational Pareto efficient points

Market allocation

p^x, p^y

Household 1:Choose X^1, Y^1, to maximize $U^1(X^1,Y^1)$ subject to

$p^xX^1 + p^yY^1 = p^x \bar{X}^1 + p^y\bar{Y}^1 = B^1$

budget constraint is a straight line passing through the endowment point $(\bar{X}^1, \bar{Y}^1)$ with slope $-\frac{p^x}{p^y}$.

Lagrangian

$L = U^1(X^1,Y^1) - \lambda\ [p^x X^1 + p^y Y^1 - B^1]$

$$\frac{\partial L}{\partial X} = \frac{\partial U^1}{\partial X^1} - \lambda p^x = 0$$

$$\frac{\partial L}{\partial Y} = \frac{\partial U^1}{\partial Y^1} - \lambda p^y = 0$$

Therefore, at the utility optimum subject to budget constraint we have

$$MRS^1_{xy} = \frac{\frac{\partial U^1}{\partial X^1}}{\frac{\partial U^1}{\partial Y^1}} = \frac{p^x}{p^y}$$; Similarly for household 2,

$$MRS^2_{xy} = \frac{\frac{\partial U^2}{\partial X^2}}{\frac{\partial U^2}{\partial Y^2}} = \frac{p^x}{p^y} \quad .$$

Equilibrium prices: p^{*x} and p^{*y} so that

$$X^{*1} + X^{*2} = \bar{X}^1 + \bar{X}^2 \equiv \bar{X}$$

$$Y^{*1} + Y^{*2} = \bar{Y}^1 + \bar{Y}^2 \equiv \bar{Y} \quad ,$$

(market clearing)

where X^{*i} and Y^{*i} , i =1, 2, are utility maximizing mix of X and Y at prices p^{*x} and p^{*y} .

$$-\frac{\partial Y^1}{\partial X^1}\Big|_{U^1=U^{1*}} = \frac{\frac{\partial U^1}{\partial X^1}}{\frac{\partial U^1}{\partial Y^1}} = \frac{p^x}{p^y}$$

$$\frac{p^x}{p^y} = \frac{\frac{\partial U^2}{\partial X^2}}{\frac{\partial U^2}{\partial Y^2}} = -\frac{\partial Y^2}{\partial X^2}\Big|_{U^2=U^{2*}}$$

The price system <u>decentralizes</u> the efficient allocation decision.